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ON THE STABILITY OF FLOWS APPEARING AT THE DISINTEGRATION OF AN ARBITRARY DISCONTINUITY

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The behavior of small perturbations is investigated in the region bounded by plane discontinuity surfaces propagating in a medium at constant velocities in opposite directions. A surface of contact discontinuity is present in the general case between the two discontinuity surfaces. Characteristics of the unperturbed motion between discontinuity surfaces are assumed constant and the perturbed flow is assumed one-dimensional. The investigation of stability of flows which occur at the disintegration of the initial discontinuity in a two-parameter medium with arbitrary equations of state reduce in many instances to such problem. The derived results are independent of the specific nature of the discontinuity surfaces (which, for example, may be detonation waves). The proposed investigation method can be applied also to cases in which more than one surface reflecting perturbations are formed on one or both sides of the initial discontinuity plane.

The problem of disintegration of an arbitrary discontinuity in a perfect gas was first solved by Kochin [1]. Let ξ be the Lagrangian coordinate of particles lying in a single plane parallel to the initial discontinuity plane, and let the plane $\xi = 0$ coincide at an instant $t = 0$ with the initial discontinuity plane. The disintegration of an arbitrary discontinuity in a perfect gas and, generally, in any medium for which the derivative $(\partial^2 p / \partial V^2)_s$, where p is the pressure, V the specific volume, and s the entropy, is positive and the derivative $(\partial p / \partial V)_s$ is continuous on each side of the plane $\xi = 0$, results in the propagation of single shock or centered rarefaction waves from each side of plane $\xi = 0$. The propagation velocity of the shock wave over particles ahead of the front is higher and behind it lower than the related speed of sound. The forward and rear boundaries of the centered wave, which are weak discontinuities, propagate over particles at sonic velocity [1, 2].

In media, for which the derivative $(\partial^2 p / \partial V^2)_s$ is of a different sign in different regions of the Vp-plane, rarefaction shocks and centered compression waves, as well as combined centered compression or rarefaction waves may be generated besides shock waves (compression shocks) and centered rarefaction waves [3]. When the derivative $(\partial p / \partial V)_s$ is continuous, only one wave of the above kind propagates from each side of the plane $\xi = 0$. As in the case of $(\partial^2 p / \partial V^2)_s > 0$, expanding regions $0 \leqslant \xi \leqslant \xi_1(t)$ and $\xi_2(t) \leqslant \xi \leqslant 0$, where the flow parameters are constant, adjoin the surface $\xi = 0$. The boundaries $\xi = \xi_1(t)$ and $\xi = \xi_2(t)$ of these regions are either strong or weak discontinuities, and surface $\xi = 0$ can be a contact discontinuity.

Qualitatively the pattern of motion which takes place at the disintegration of an arbitrary discontinuity may increase in complexity, if the motion is accompanied by chemical reactions [4] or phase transformations, or when the medium interacts with an electromagnetic field [5]. For example, in some cases two discontinuities which reflect perturbations and propagate in the same direction may be formed on one or two sides of the surface $\xi = 0$. However in a number of cases the investigation of flow stability reduces to the problem considered below.

1. We assume that the equations of state for the medium conform to known constraints of thermodynamic character and consider only cases, when either a single compression or rarefaction shock, or a single conventional or combined rarefaction or compression waves propagate from both sides of the plane $\xi = 0$, without imposing on the medium properties (in particular on derivatives $(\partial^2 p / \partial V^2)_s$ and $(\partial p / \partial V)_s$) any further limitations.

According to [3] the stability is defined in such cases by the behavior of perturbations in the region $\xi_2(t) \leqslant \xi \leqslant \xi_1(t)$. Perturbations outside that region can be assumed to be zero.

The equations of state of certain media admit transformations for which the laws of conservation and the requirement for nondecreasing entropy are satisfied, and which are, nevertheless, considered unattainable. In particular, nonevolutionary shocks and shocks for which the stability conditions for an isolated shock wave are not satisfied, belong to such shocks [6]. For shocks not excluded by these constraints, the following inequalities must be satisfied:

$$0 \leqslant \mu < 2(1 + M), \quad M \leqslant 1, \quad M_0 \geqslant 1 \tag{1.1}$$

$$\mu = 1 + j^2 \left(\frac{\partial V}{\partial p}\right)_H, \quad M = |j| \frac{V}{a}, \quad j^2 = \frac{p - p_0}{V_0 - V}$$

where $(\partial V / \partial p)_H$ is a derivative along the shock adiabate, a is the speed of sound, and the subscript zero denotes parameters ahead of the shock front.

2. Let the parameters related to the disintegration of the flow discontinuity be subjected at the instant t_0 in the region $\xi_2(t_0) \leqslant \xi \leqslant \xi_1(t_0)$ to small perturbations, while outside that region initial perturbations are absent. The last assumption is not a basic limitation and, as shown in [3], is acceptable in cases in which surface $\xi = \xi_r(t)$ is the rear boundary of a conventional or centered combination wave.

We assume that the perturbed flow is adiabatic and restrict our investigation to the behavior of perturbations of velocity u' and pressure p'. As a function of ξ, the entropy perturbations in region $\xi_2(t_0) \leqslant \xi \leqslant \xi_1(t_0)$, $t > t_0$ are determined by the input data, while in regions $\xi_2(t) \leqslant \xi \leqslant \xi_2(t_0)$ and $\xi_1(t_0) \leqslant \xi \leqslant \xi_1(t)$ they are determined by the perturbed entropy for $\xi = \xi_2(t) + 0$ and $\xi = \xi_1(t) - 0$. The latter, as well as perturbations j_1' and j_2' are determined by the linearized conditions at a normal shock for known u' and p'.

We seek velocity and pressure perturbations in the form $u' = V(I^+ - I^-) / 2a$ and $p' = (I^+ + I^-) / 2$; here and subsequently V and a denote, respectively, the specific volume and the speed of sound in the unperturbed flow. The unperturbed flow parameters and the quantities $I^\pm$ are denoted in region $0 \leqslant \xi \leqslant \xi_1(t)$ by subscript 1 and in region $\xi_2(t) \leqslant \xi \leqslant 0$ by subscript 2. In a linearized formulation we obtain for I^+ and I^- the following system of equations and initial and boundary conditions:

$$\frac{\partial I_r^{\pm}}{\partial t} \pm \frac{a_r}{V_r}\frac{\partial I_r^{\pm}}{\partial \xi} = 0, \quad r = 1, 2 \tag{2.1}$$

$$t = t_0, \ I_1^{\pm} = I_{01}^{\pm}(\xi), \ I_2^{\pm} = I_{02}^{\pm}(\xi)$$

$$\xi = 0, \ I_1^+ = (1 - A) I_2^+ + A I_1^-, \ I_2^- = -A I_2^+ + (1 + A) I_1$$

$$\xi = \xi_1(t) \equiv j_1 t, \ I_1^- = K_1 I_1^+; \ \xi = \xi_2(t) \equiv j_2 t, \ I_2^+ = K_2 I_2^-$$

where ξ is a Lagrangian coordinate related to the Euler coordinate x by formula $(\partial x / \partial \xi) = V$, A and K_r are the coefficients of perturbation reflection from the contact discontinuity and from surface $\xi = \xi_r(t)$, respectively, along which the boundary conditions are obtained by linearizing relationships at the normal shock. In the case of a shock wave for A in (2.1) and for K_r we have

$$A = \frac{(a/V)_2 - (a/V)_1}{(a/V)_2 + (a/V)_1}, \qquad K_r = -\frac{2(1 - M_r) - \mu_r}{2(1 + M_r) - \mu_r} \tag{2.2}$$

It follows from (2.2) that $0 \leqslant |A| < 1$, if $a_r \neq 0$ and are bounded. For $A = 0$ the surface $\xi = 0$ does not reflect perturbations and the amplitude of transmitted waves remains unaltered. In considering the stability criterion for $A = \pm 1$, we shall bear in mind that the question is about the motion stability of a medium which occupies the half-space $\xi \geqslant 0$ (or $\xi \leqslant 0$) when the boundary condition at surface $\xi = 0$ is of the form $I^+ = I^-$ $(A = 1)$ or $I^+ = -I^-$ $(A = -1)$. The first of these corresponds to the disintegration of the initial discontinuity at the rigid wall bounding the medium, and the second to disintegration at the free boundary. It should also be noted that, when surface $\xi = \xi_r(t)$ is either a weak dicontinuity or shock at which $M_r = 1$, then $\mu_r = 0$ and $K_r = 0$.

Since the system of equations for perturbations contains invariants, hence a variation of perturbation amplitude can only take place at interaction with discontinuity surfaces.

The conditions of flow stability are readily established in particular cases in which amplitude variation occurs only at interaction with two surfaces. To do so it is sufficient to observe the variation of perturbation amplitude for a single successive reflection from these surfaces. In such particular cases the criteria of asymptotic and indifferent stabilities, and of instability of the flow are, respectively, of the form

$$\begin{aligned} &A = 0: \ |K_1K_2| < 1, \ |K_1K_2| = 1, \ |K_1K_2| > 1 \\ &K_r = 0: \ |AK_q| < 1, \ |AK_q| = 1, \ |AK_q| > 1, \ q \neq r \\ &A = \pm 1: \ |K_r| < 1, \ |K_r| = 1, \ |K_r| > 1 \end{aligned} \tag{2.3}$$

In the last case $r = 1$ $(r = 2)$ when the medium occupies the half-space $\xi \geqslant 0$ (half-space $\xi \leqslant 0$).

In the general case, when the reflection coefficients K_1 and K_2 are nonzero and $0 < |A| < 1$, the behavior of perturbations is more complex, and to formulate conditions of flow stability a more detailed analysis based on an analytic presentation of solution is necessary.

3. The analytic representation of solution of problem (2.1) can be obtained by methods of operational calculus, using new independent variables $\tau = \ln(t / t_0)$ and $\eta = \xi / j_r t$. Another method is also possible, if one takes into account that owing to the properties of invariants I_r^+ and I_r^- it is sufficient in investigations of the considered flow stability to analyze their asymptotics at the contact discontinuity.

Note that acoustic waves which leave the contact discontinuity at one and the same time, return, after a single reflection from both shocks, to the contact discontinuity at the same instant of time (Fig. 1). The instants of time shown in Fig. 1 are defined by the relationships

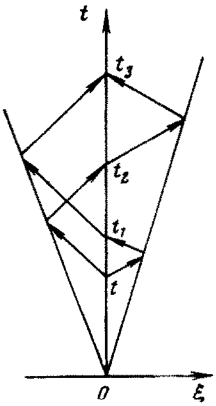

Fig. 1

$$t_1 = t \,/\, B_1,\ \ t_2 = t \,/\, B_2,\ \ t_3 = t \,/\, B_1 B_2$$
$$(B_r = (1 - M_r)\,/\,(1 + M_r),\quad r = 1,\ 2)$$

Using the above property together with boundary conditions (2.1) it is possible to show that each of the invariants $I_r^{\pm}$ must satisfy for $\xi = 0$ and $t > t_0$ one and the same functional equation

$$I\,(\tau + \alpha + \beta) + AK_1 I\,(\tau + \beta) - AK_2\,I\,(\tau + \alpha) - K_1 K_2\,I\,(\tau) = 0 \qquad (3.1)$$
$$\tau = \ln\,(t\,/\,t_0),\ \ \alpha = -\ln B_1,\ \ \beta = -\ln B_2$$

Invariant $I\,(\tau)$ may be considered to be a known function of initial data in the interval $0 \leqslant \tau \leqslant \alpha + \beta$. It is possible to show by methods of operational calculus that the solution of Eq. (3.1) is of the form

$$I\,(\tau) = \Psi\,(\tau)\,\Phi\,(0) + \int_0^{\tau} \Phi'\,(\tau - t)\,\Psi\,(t)\,dt$$

where $\Phi\,(\tau)$ is a periodic function whose fundamental period is equal $\alpha + \beta$. In the interval $0 \leqslant \tau \leqslant \alpha + \beta$ function $\Phi\,(\tau)$ is expressed in terms of initial data ; function $\Phi\,(\tau)$ can be made differentiable by suitably ordering these data. Function $\Psi\,(\tau)$ is represented in the form of the complex integral

$$\Psi\,(\tau) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{e^{(\alpha+\beta)z} - 1}{z\varphi\,(z)}\, e^{\tau z} dz \qquad (3.2)$$

$$\varphi\,(z) = e^{(\alpha+\beta)z} + AK_1 e^{\beta z} - AK_2 e^{\alpha z} - K_1 K_2$$

whose integration is carried out in the plane of the complex variable z along the straight line $\operatorname{Re} z = \sigma_0$ which lies to the right of all zeros of function $\varphi\,(z)$.

In the special cases in which α and β are commensurable the integral (3.2) is computed with the use of Laplace transformation tables (see e.g., [7]), and for functions $\Psi\,(\tau)$ and $I\,(\tau)$ simple formulas are obtained. Let $m\alpha = l\beta$, where m and l are relatively prime integers, and let function $y = g\,(t)$ be the solution of the ordinary differential equation

$$y^{(m+l)}\,(t) + AK_1 y^{(m)}\,(t) - AK_2\, y^{(l)}\,(t) - K_1 K_2\, y\,(t) = 0 \qquad (3.3)$$

and satisfy the initial condition

$$y\,(0) = y'(0) = \ldots . \ y^{(m+l-2)}\,(0) = 0,\ y^{(m+l-1)}(0) = 1 \qquad (3.4)$$

Then functions $\Psi\,(\tau)$ and $I\,(\tau)$ are expressed in terms of the following derivatives of function $g\,(t)$ computed for $t = 0$:

$$\Psi(\tau)=\sum_{r=0}^{m+l-1} g^{(n+r)}(0),\ I(\tau)=\sum_{r=0}^{m+l-1}\Phi\left(\tau-\frac{n+r+1}{l}\alpha\right)g^{(n+r)}(0)\ (n\geqslant m+l)$$

where $n = n(\tau)$ is an integer which satisfies the inequality

$$n(\tau)\leqslant \tau\, l/\alpha < n(\tau)+1$$

The asymptotic behavior of function $I(\tau)$ depends on the distribution in the complex plane $w = e^{\alpha z/l}$ of roots λ_q of the polynomial

$$N(w) = w^{m+l} + AK_1w^m - AK_2w^l - K_1K_2$$

If all λ_q are different,

$$g(t)=\sum_{q=1}^{m+l}\frac{e^{\lambda_q t}}{N'(\lambda_q)},\qquad g^{(n+r)}(0)=\sum_{q=1}^{m+l}\frac{\lambda_q^{r+n(\tau)}}{N'(\lambda_q)}$$

and, when $|\lambda_q|\leqslant 1$, the flow is stable.

Note that for $K_1K_2\neq 0$ the polynomial $N(w)$ cannot have roots whose multiplicity exceeds 2 and double complex roots whose absolute values are unity. The polynomial $N(w)$ can have the double root $w = 1$ and $w = -1$. In these cases the flow is unstable. Obviously the values of A, K_1 and K_2 which in the space (A, K_1, K_2) correspond to multiple roots $w = \pm 1$ lie either at the boundary of stability region or in the region of instability (in the examples given in Fig. 4 these are indicated by small circles).

Thus for commensurable α and β the flow is asymptotically stable, if all roots of $N(w)$ lie inside the unit circle $|w| < 1$. As an example we adduce the asymptotic stability criterion for certain values of m and l

$$\begin{aligned}&1 + A(K_1 - K_2) - K_1K_2 > 0\\ &1 - A(K_1 - K_2) - K_1K_2 > 0 \qquad (m = l = 1)\\ &1 + K_1K_2 > 0\end{aligned}\tag{3.5}$$

$$\begin{aligned}&1 + A(K_1 - K_2) - K_1K_2 > 0\\ &1 - A(K_1 - K_2) + K_1K_2 > 0 \qquad (m = 2,\ l = 1)\\ &1 + AK_2(1 - K_1^2) - K_1^2K_2^2 > 0\end{aligned}$$

$$\begin{aligned}&1 + A(K_1 - K_2) - K_1K_2 > 0\\ &(1 + K_1K_2)^2(1 - K_1K_2) - A^2K_2(K_1 + K_2)(1 - K_1^2) > 0\\ &1 - A(K_1 - K_2) - K_1K_2 > 0 \qquad (m = 3,\ l = 1)\end{aligned}$$

4. Cases in which α and β are incommensurable is evidently of great interest. In such cases the asymptotic behavior of function $I(\tau)$ depends also on the distribution of zeros of the quasi-polynomial $\varphi(z)$.

When α/β is irrational the quasi-polynomial $\varphi(z)$ is an analytic almost periodic function in the band $-\infty < \operatorname{Re} z < \infty$. For fixed A, K_1 and K_2 there exist constants d_1 and d_2 such that all zeros of the quasi-polynomial $\varphi(z)$ lie within the band $d_2 < \operatorname{Re} z < d_1$. For $\operatorname{Re} z > d_1$ function $1/\varphi(z)$ is almost periodic and bounded [8].

If $d_1 < 0$, the considered flow is asymptotically stable. It can be shown that for $d_1 < 0$

$$|I(\tau)| \leqslant \left\{[e^{2(\alpha+\beta)|\sigma_0|} - 1] \frac{1 + A^2(K_1^2 + K_2^2)}{8\pi\sigma_0^2}\right\} Le^{\tau\sigma_0}$$

$$d_1 < \sigma_0 < 0, \quad L = \max_{\operatorname{Re} z=\sigma_0} \left|\frac{1}{\varphi(z)}\right| \max_{0\leqslant\tau\leqslant\alpha+\beta} |I'(\tau)|$$

Let σ^* be the exact upper boundary of the real parts of the quasi-polynomial $\varphi(z)$ zeros when α / β is irrational. Let us establish the criterion of the flow asymptotic stability, i.e. the relations between parameters A, K_1 and K_2 for which $\sigma^* < 0$, and consider the equation

$$e^{-(\alpha+\beta)z} \varphi(z) \equiv 1 + AK_1e^{-\alpha z} - AK_2e^{-\beta z} - K_1K_2e^{-(\alpha+\beta)z} = 0 \qquad (4.1)$$

According to Kronecker's theorem (see, e.g., [8]) it is possible for an irrational α / β to assign with any degree of accuracy to the arguments of the second and third terms of (4.1) any a priory specified set of values by a suitable choice of $\operatorname{Im} z$. The closing of the set of real parts of roots of Eq. (4.1) coincides, consequently, with the set of values of σ, which satisfy the equation

$$1 + A^2(c\zeta_1 - b\zeta_2 - cb\zeta_1\zeta_2) = 0 \qquad (4.2)$$

$$c = \left|\frac{K_1}{A}\right| e^{-\alpha\sigma}, \quad b = \left|\frac{K_2}{A}\right| e^{-\beta\sigma}, \quad \sigma = \operatorname{Re} z, \quad \zeta_r = e^{-i\theta_r}$$

where θ_r $(r = 1, 2)$ are arbitrary real numbers. Let us write Eq. (4.2) in the form

$$\zeta_1 = \frac{1 - A^2b\zeta_2}{A^2c(b\zeta_2 - 1)} \qquad (4.3)$$

Since ζ_1 and ζ_2 can assume any arbitrary independent values along the unit circle, the set of all σ and ν $(\nu = \cos\theta_2)$, for which the modulus (R) of the right-hand part of (4.3) is equal unity, represents the solution of Eq. (4.3). It follows from (4.3) that

$$R = \frac{1}{A^2c}\left[\frac{1 + (A^2b)^2 - 2A^2b\nu}{1 + b^2 - 2b\nu}\right]^{1/2}$$

Let us point out some of the properties of function $R(\sigma, \nu)$ required in the subsequent analysis. We denote by σ_r $(r = 1, 2, 3)$ the values of σ for which $b(\sigma_r) = |A|^{r-3}$. Having computed the derivatives of function $R(\sigma, \nu)$, we readily find that $(\partial R / \partial\sigma) > 0$ when $\nu = -1$ and for $\nu = 1$ and $\sigma_1 \leqslant \sigma < \sigma_3$, while

$$\frac{\partial R}{\partial \nu} \begin{cases} < 0, & -\infty < \sigma < \sigma_2 \\ = 0, & \sigma = \sigma_2 \\ > 0, & \sigma_2 < \sigma < \infty \end{cases}$$

We point out that $R(\sigma_1, 1) = 0$. The described properties of function $R(\sigma, \nu)$ are sufficient for deriving the required statements (curves of functions $R(\sigma, 1)$ and $R(\sigma, -1)$ are shown in Fig. 2 for illustration).

It is obvious that, when $\sigma_1 \geqslant 0$, the roots of Eq. (4.3) always lie in right-hand half-plane. Let $\sigma_1 < 0$ and $\sigma_2 > 0$. In that case $\min_{\sigma\geqslant 0} R(\sigma, \nu) = R(0, 1)$. If $\sigma_2 \leqslant 0$, then $\min_{\sigma\geqslant 0} R(\sigma, \nu) = R(0, -1)$. Hence $\sigma^* < 0$ when the following conditions are satisfied: $R(0, 1) > 1$ for $1 < |K_2| < 1 / |A|$, while for $|K_2| \leqslant 1$ we have $R(0, -1) > 1$.

This implies that for irrational α / β the flow is asymptotically stable, if the coefficients A, K_1 and K_2 belong to the region defined by the inequalities

$$1 + |A|(|K_1| - |K_2|) - |K_1K_2| > 0 \tag{4.4}$$
$$1 - |A|(|K_1| - |K_2|) - |K_1K_2| > 0$$

The set of points $P(A, K_1, K_2)$ which satisfy the conditions of asymptotic stability (4.4), and in particular cases those defined by (2.3), constitutes region Q in the space of parameters A, K_1 and K_2. A general form of that region is shown in Fig. 3, and

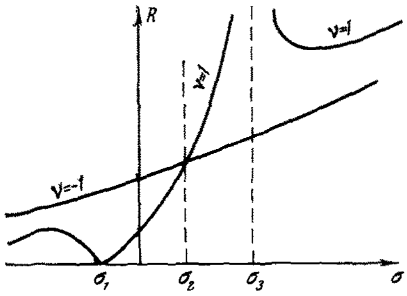

Fig. 2

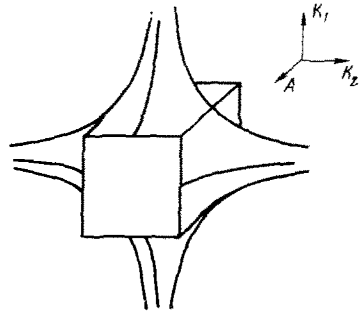

Fig. 3

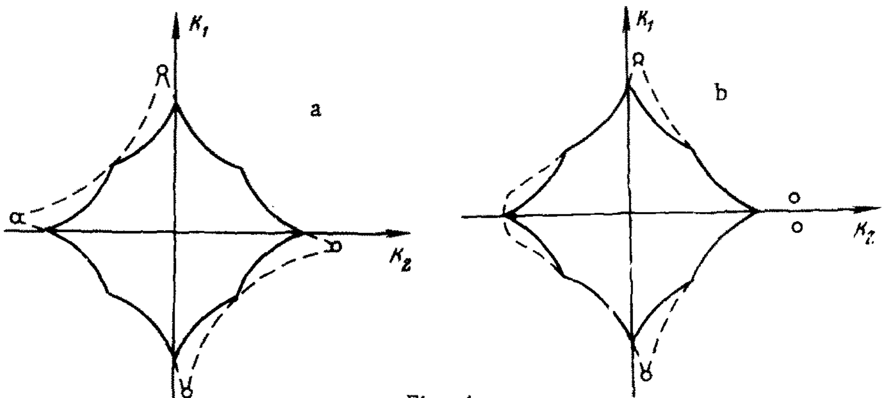

Fig. 4

its boundaries in the plane $A = A_1$ $(0 < A_1 < 1)$ appear in Fig. 4. The solid line corresponds to irrational α / β. For comparison the boundaries of region Q for $\alpha = \beta$ and $2\alpha = \beta$ are also shown in Fig. 4 (a) and (b), respectively. The conditions of asymptotic stability are defined by (3.5). For rational α / β the boundaries of those parts of region Q, where they diverge from the boundaries of region Q in the case of irrational α / β, are shown by dash lines.

It is necessary to stress that conditions (2.3), (3.5) and (4.4) were obtained on the assumption that the shocks present in the stream are evolutional $(\mu_r \neq 2(1 + M_r)$, $M_r \leqslant 1$, $M_{0r} \geqslant 1)$, and that the first of the inequalities (1.1) was not used in their derivation. On the other hand, the conditions of flow stability (3.5), (4.4), and those defined by (2.3) admit shocks which are unstable relative to small nonunivariate perturbations. At such shock $\mu < 0$ or $\mu > 2(1 + M)$ [6]. Flows with such shocks must be considered impossible in reality.

In other words the considered flow is stable, if inequalities (1.1) and the condition derived in Sects. 2–4 which relates to the particular flow are satisfied at the shocks.

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ON THE PROBLEM OF OPTIMIZATION OF THE SHAPE OF A BODY IN A VISCOUS FLUID

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We obtain necessary conditions for minimum drag of a body in a viscous fluid when the flow is described either by the exact Navier-Stokes equations or by the approximate Oseen equations. We study some of the characteristics of optimal bodies. The problem of optimizing the shape of a body in the flow of a viscous fluid was considered previously in [1] in the Stokes approximation, wherein necessary conditions were derived which the shape of a body of minimum drag must satisfy; some qualitative characteristics of optimal shapes were also investigated.

1. The stationary flow of a viscous incompressible fluid over a body S is described by the Navier-Stokes equations and the no-slip boundary conditions on the body surface. For convenience in our transformations we consider, in the sequel, a finite volume of fluid Ω, bounded in its interior by the surface of a body S and, on the outside, by a surface Σ on which the velocity vector $\mathbf{u}$ is specified. For the case in which an unbounded mass of fluid flows over the body the minimum distance from the body surface S to the surface Σ must tend to infinity.